

THE ALGEBRA $H^2(S)$

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By a well-known theorem [3] p. 245, if $F(t)$ is locally integrable on $(-\infty, \infty)$ and if

$$(i) \int_0^{\infty} e^{-2\sigma t} |F(t)|^2 dt < \infty \text{ and}$$

(ii) $\int_{-\infty}^0 e^{-2\eta t} |F(t)|^2 dt < \infty$ for $\sigma < \eta$, then $\int_{-\infty}^{\infty} e^{-zt} F(t) dt$ converges mean-square for $\sigma \leq R(z) \leq \eta$.

Further, for $\sigma < R(z) < \eta$ the integral converges absolutely and hence defines an analytic function and finally

$$\int_{-\infty}^{\infty} e^{-2xt} |F(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |f(x + iy)|^2 dy, \quad \sigma < x < \eta.$$

Let S be the strip $\{z | \sigma < R(z) < \eta, -\infty < \sigma < \eta < \infty\}$ and

$$(iii) H^2(S) = \left\{ f \mid f \text{ analytic in } S, \|f\|_S = \sup_{\sigma < x < \eta} \left[\int_{-\infty}^{\infty} |f(x + iy)|^2 dy \right]^{\frac{1}{2}} < \infty \right\}$$

then it is clear that $H^2(S)$ contains the Laplace transforms of functions on $(-\infty, \infty)$ which satisfy conditions (i) and (ii). $H^2(S)$ is seen to be closed under point-wise addition and scalar multiplication. This space would be more interesting if it were closed under point-wise multiplication and complete with respect to the norm $\|f\|_S$ and hence a Banach Algebra. To achieve both of these replace (iii) by

$$H^2(S) = \left\{ f \mid f \text{ analytic for } \sigma - \epsilon < R(z) < \eta + \epsilon, \right.$$

$$f \text{ cont. for } \sigma - \epsilon \leq R(z) \leq \eta + \epsilon,$$

$$\left. \sup \left[\int_{-\infty}^{\infty} |f(x + iy)|^2 dy \right]^{\frac{1}{2}} < \infty, \sigma - \epsilon \leq x \leq \eta + \epsilon \right\}.$$

* Partially supported by NSF Grant GP-1591.

It now follows that for $f \in H^2(S)$, there exists M_f such that $|f(x + iy)| < M_f$ for $\sigma - \epsilon \leq R(z) \leq \eta + \epsilon$. By the Cauchy-Schwartz Inequality, then $f, g \in H^2(S)$ implies $fg \in H^2(S)$. If $\|f\|_S$ is used as in (iii) then the space is complete. In [2] the author has shown that on the subspace of Laplace Transforms as per the above theorem in [3], the norm topology is stronger than the topology of uniform convergence on compact subsets of S . Our interest in this space is twofold, because of its relation to Laplace Transforms, and the relative strength of these topologies and the role of these analytic functions played in representing generalized functions or distributions [1] and also simply because it has not been previously investigated as extensively as the well-known H^p spaces.

We proceed now to determine some of the structure of the algebra, in particular proper ideals.

Let $I_{z_0} = \{f \in H^2(S), z_0 \in S, f(z_0) = 0\}$. Then it is clear that for each $z \in S$, I_z is an ideal. Let $u \in H^2(S)$, $u(z_0) \neq 0$, without loss of generality assume $u(z_0) = 1$; then $ug - g \in I_{z_0}$ for all $g \in H^2(S)$ and hence u is a relative identity for I_{z_0} . Since $H^2(S)$ is commutative, u is both a left and right relative identity and in fact I_{z_0} is a regular maximal ideal; because

$$\bigcap_{z \in S} I_z = \{0\}$$

the algebra is semi-simple.

An example of an ideal which is not known to be proper is D ,

$$D = \left\{ f \left\| \left\| \frac{e^{hz} f(z) - f(z)}{h} - z f(z) \right\|_S \rightarrow 0 \text{ as } h \rightarrow 0 \right\} \right\}.$$

Under the weaker topology of uniform convergence on compact subsets of S it is known that

$$\frac{e^{hz} f(z) - f(z)}{h} - z f(z) \rightarrow 0 \text{ as } h \rightarrow 0$$

for all f , analytic in S [1]. For those functions satisfying conditions (i) and (ii) and whose first derivatives satisfy the same conditions, their Laplace transforms are clearly contained in D . It seems likely that D is proper and not regular.

Other questions that we may ask about $H^2(S)$ include the following:

(1) For what subspace, $M \subseteq H^2(S)$, is the mapping $f(z) \rightarrow z f(z)$ defined and what is the invariant subspace of this mapping?

(2) It is known that the algebra of all functions analytic in S is separable under the topology of uniform convergence on compact subsets. Is $H^2(S)$ separable?

(3) The mapping $f \rightarrow f^*$ is an involution on $H^2(S)$, where $f^*(z) = \overline{f(\bar{z})}$. For those f that are Laplace transforms, however, $f = f^*$. What is the invariant subspace of $f \rightarrow f^*$?

(4) Are all the regular maximal ideals of $H^2(S)$ realizable as kernels of homomorphisms of the form

$$\hat{z}(f) = f(z), \quad z \in S?$$

REFERENCES

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- [2] ———, Topologies for Laplace transform spaces, (to appear in the *Pacific Journal of Math.*).
- [3] D. V. WIDDER, *The Laplace Transform*, Princeton, (1946).